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Frequency Optimization with Respect to Lumped Mass Position

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The sensitivity analysis of a natural frequency with respect to the change of a lumped mass position has been a research topic for some time. However, relatively little work has been done on optimizing the lumped mass position. The frequency sensitivity with respect to both the position and magnitude of a lumped mass is investigated by the discrete method. Furthermore, an evolutionary algorithm is presented for frequency optimization of beam and plate structures to positions of the lumped mass attachments, which can move in the structural domain. Based on the design sensitivity analysis, the most efficient lumped mass is shifted to optimize the frequencies asymptotically. Three model examples are provided to demonstrate the validity of the sensitivity analysis and the effectiveness of the optimization method.

Nomenclature

a	=	x coordinate of a lumped mass
b	=	y coordinate of a lumped mass
D	=	design region
E	=	Young's modulus
$[K]$, $[M]$	=	global stiffness and mass matrices, respectively
k	=	number of the movable lumped masses
L	=	length of the beam
M	=	attached lumped mass
$[M]_L$	=	equivalent mass matrix of a lumped mass
N	=	shape function of an element
n	=	number of the constrained natural frequencies
T	=	kinetic energy
v	=	transverse displacement of a beam
w	=	transverse displacement of a plate
α, β	=	angles of a specific direction to the x and y axes, respectively
$\{\delta\}$	=	column vector of element nodal degrees of freedom
θ	=	slope of a node
ν	=	Poisson ratio
ρ	=	material density
$\{\phi\}$	=	vibration mode
Ω	=	set of frequency limitations
ω	=	(circular) natural frequency

Subscripts

i	=	ith component
$,x$	=	$\partial(\,)/\partial x$

I. Introduction

RECENTLY, sensitivity analysis of a natural frequency with respect to the position of a lumped (nonstructural) mass has attracted much attention.[1,2] Significant advances have been achieved with results from the existing literature. Wang[1] has applied the classical normal mode method to derive the frequency sensitivity of an Euler–Bernoulli beam with respect to a lumped mass position by treating the lumped mass as an external excitation imposed on a restrained structure. Oguamanam et al.[2] have derived the closed-form frequency sensitivity of a beam and a Reissner–Mindlin bending plate using the generalized Rayleigh quotient of the system in conjunction with Rayleigh's principle of stationary values. With the formulas therein, one can predict the effect of a lumped mass movement on the structural dynamic characteristics without solving an eigenvalue problem for every potential change of the lumped mass position.

Structural optimization with fixed positions of lumped masses has been widely and thoroughly investigated with various methods. For dynamic optimization, structural topology, configuration, and sizing, such as element thickness or cross-sectional area, are generally referred to as design variables for weight minimization or fundamental frequency maximization of a structure.[3–5] In frequency optimization problems, the positions of lumped masses can also be considered as design variables. These lumped masses may represent the fuel, payload, attachments, and even dynamic balances, etc. They are of great importance in many practical situations.[3] Moreover, it is well known that the lumped mass position can influence the structural frequency significantly. However, the efforts in such a field, to the best of our knowledge, are limited, and little work is available at present.

In this paper, we assume that the lumped mass or masses already exist in the structural system. For simplicity, only the transverse inertia of each lumped mass will be accounted for, whereas its effect of the rotatory inertia will be ignored. The objectives of this study are twofold. First, the design sensitivity of a natural frequency with respect to a lumped mass position is derived for beam and plate structures within the context of the finite element (FE) method. Because the structural dynamic analysis resorts most frequently to the numerical implementation with the FE method, such a derivation of the design sensitivity is compatible with the common modal analysis numerically. By means of the element shape functions, the closed-form sensitivity formulas are derived straightforwardly. The present results are identical to those achieved with other methodologies.[1,2] In addition, the frequency sensitivity to the magnitude of a lumped mass is also described for the purpose of completeness. Second, an evolutionary optimization procedure is proposed for various frequency optimization problems based on the sensitivity information. This heuristic algorithm works well for discrete design variables and has been successfully applied to structural topology and truss shape optimization problems.[6,7] Positions of the lumped masses will be moved to optimize the natural frequencies of a structure. In this study, the lumped mass will always be assumed to be located at the nodes of the structural FE model, which, therefore, implies that it moves along the elementary edges and that the move interval (shift step) is the element size. Thus, the optimal design may depend slightly on the mesh density with the fixed grid mesh scheme. If required, a fine FE model in a local region can be utilized to acquire a more accurate solution. Through the lumped mass relocations, the constrained frequencies will be optimized gradually with

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the minimum movement of lumped masses because the structural center of mass is an important consideration for practical designs of aerospace vehicles. On the basis of design sensitivity analysis, the most efficient lumped mass is identified and shifted systematically in the desired directions. The method is illustrated with three typical structures and is shown to be quite effective.

II. Frequency Sensitivity Analysis with Respect to Lumped Mass Position

In dynamic analysis, the generalized eigenvalue problem of an undamped structure is represented by

$$\left([K] - \omega_i^2[M]\right)\{\phi\}_i = \{0\} \tag{1}$$

where ω_i is the ith (circular) natural frequency and $\{\phi\}_i$ is the associated vibration mode of the structure, which has been orthogonally normalized to yield unit modal mass in this context. Note that, in the present investigation, ω_i^2 is an implicit function of the lumped mass position, that is, it can not be expressed explicitly with the position parameter.

As is well known, the design sensitivity analysis aims at finding the effects of design variables on natural frequencies. It plays a vital role in the optimality criteria algorithm. Generally, the sensitivity of the ith natural frequency with regard to the position of a lumped mass can be evaluated from[3]

$$\frac{\mathrm{d}\omega_i^2}{\mathrm{d}a} = \{\phi\}_i^T\left(\frac{\mathrm{d}[K]}{\mathrm{d}a} - \omega_i^2\frac{\mathrm{d}[M]}{\mathrm{d}a}\right)\{\phi\}_i \tag{2}$$

where a is the coordinate of a lumped mass. Although the movement of a lumped mass would not influence the stiffness matrix of the structure, it will inevitably lead to the mass redistribution in the system and then change the structural natural frequency.

A. Beam Element with a Lumped Mass Attachment

Consider a uniform beam element of length L with a lumped mass attached in its span, as shown in Fig. 1. The element is modeled based on the classical Euler–Bernoulli beam theory. The transverse displacement of the lumped mass M can be represented approximately with the nodal displacement quantities of the element: transverse displacements and slopes:

$$v(a) = [N_1 \quad N_2 \quad N_3 \quad N_4]_{(a)} \cdot \begin{Bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{Bmatrix} \tag{3}$$

where N_{1-4} are shape functions of the beam element, which should be independent of the boundary conditions. Therefore, the modal kinetic energy T of the lumped mass is expressed as

$$T = \frac{1}{2}Mv^2(a) = \frac{M}{2}\{v_1 \quad \theta_1 \quad v_2 \quad \theta_2\} \times \begin{bmatrix} N_1^2 & N_1N_2 & N_1N_3 & N_1N_4 \\ & N_2^2 & N_2N_3 & N_2N_4 \\ & & N_3^2 & N_3N_4 \\ \text{Sym.} & & & N_4^2 \end{bmatrix}_{(a)} \begin{Bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{Bmatrix} \tag{4}$$

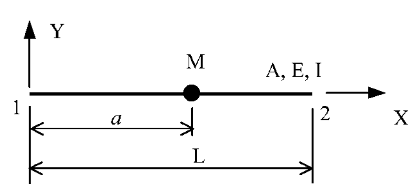

Fig. 1 Beam element with a lumped mass attachment.

Then the equivalent mass matrix of the lumped mass $[M]_L$ is given as

$$[M]_L = M\begin{bmatrix} N_1^2 & N_1N_2 & N_1N_3 & N_1N_4 \\ & N_2^2 & N_2N_3 & N_2N_4 \\ & & N_3^2 & N_3N_4 \\ \text{Sym.} & & & N_4^2 \end{bmatrix}_{(a)} \tag{5}$$

Because the movement of a lumped mass does not influence the element mass matrix, the sensitivity of the ith natural frequency with respect to the position of a lumped mass can be obtained according to Eq. (2):

$$\frac{\mathrm{d}\omega_i^2}{\mathrm{d}a} = -\omega_i^2\{\phi_e\}_i^T\frac{\mathrm{d}[M]_L}{\mathrm{d}a}\{\phi_e\}_i \tag{6}$$

where $\{\phi_e\}_i$ is the ith mode shape of the related beam element, which contains only the corresponding entries of $\{\phi\}_i$. Normally, the Hermite shape functions are adopted for the Euler–Bernoulli beam element[8]:

$$N_1 = 1 - 3(a/L)^2 + 2(a/L)^3, \qquad N_2 = a - 2L(a/L)^2 + L(a/L)^3$$

$$N_3 = 3(a/L)^2 - 2(a/L)^3, \qquad N_4 = -L(a/L)^2 + L(a/L)^3 \tag{7}$$

Note the assumption that the lumped mass is attached only at one of the two ends of the beam element. Then, the derivative of the mass matrix of the lumped mass is either

$$\left.\frac{\mathrm{d}[M]_L}{\mathrm{d}a}\right|_{a=0} = \begin{bmatrix} 0 & M & 0 & 0 \\ M & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

or

$$\left.\frac{\mathrm{d}[M]_L}{\mathrm{d}a}\right|_{a=L} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & M \\ 0 & 0 & M & 0 \end{bmatrix} \tag{8}$$

Substitution of the preceding expressions into Eq. (6) yields the sensitivity of the ith natural frequency, respectively:

$$\left.\frac{\mathrm{d}\omega_i^2}{\mathrm{d}a}\right|_{a=0} = -2\omega_i^2 v_{1i}\theta_{1i}M, \quad \text{or} \quad \left.\frac{\mathrm{d}\omega_i^2}{\mathrm{d}a}\right|_{a=L} = -2\omega_i^2 v_{2i}\theta_{2i}M \tag{9}$$

where v_{1i} and θ_{1i} denote the transverse displacement and the slope of the ith vibration mode, respectively, at end 1 of the beam element, and v_{2i} and θ_{2i} are the corresponding items at end 2.

Because the continuity of the nodal displacement and slope is always imposed between two neighboring beam elements, it can be recognized that the sensitivities obtained with Eq. (9) are the same at any node from two neighboring elements. Thus, the subscripts symbolically indicating the element end in Eq. (9) will be eliminated subsequently, and the nodal displacement quantities are just those of the lumped mass.

For an Euler–Bernoulli beam, the following relationship between the transverse displacement and slope exists:

$$\theta = v' = \frac{\mathrm{d}v}{\mathrm{d}x} \tag{10}$$

Therefore, the sensitivity of the ith natural frequency to the lumped mass position is easily obtained:

$$\frac{\mathrm{d}\omega_i^2}{\mathrm{d}a} = -2\omega_i^2 M v_i(a) v_i'(a) \tag{11}$$

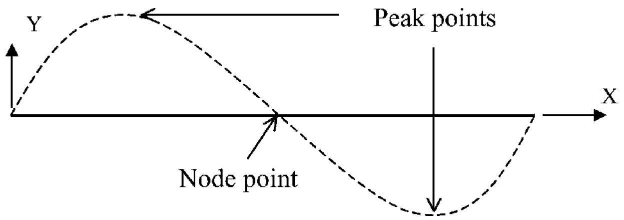

Fig. 2 Schematic diagram of a mode shape.

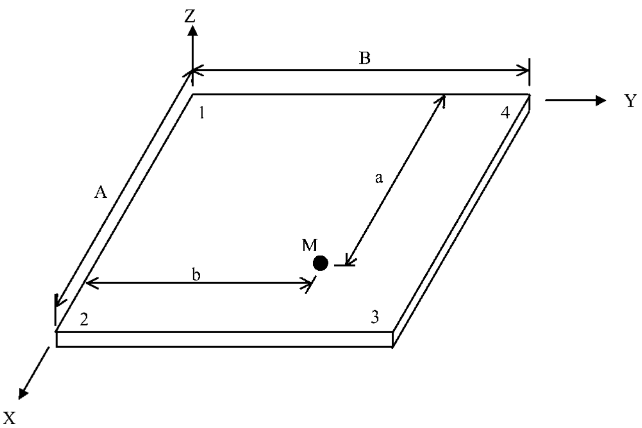

Fig. 3 Thin-plate element with a lumped mass attachment.

where $v_i(a)$ and $v_i'(a)$ are the transverse displacement and its derivative of the beam, evaluated at the position of the lumped mass attachment, respectively. This result is the same as Eq. (9) given in Ref. 2, whereas the present derivation is more straightforward.

Note that the product $\omega_i^2 M v(a)$ in Eq. (11) could be interpreted as the inertial force of the lumped mass according to D'Alembert's principle. Equation (11) indicates that the frequency sensitivity is proportional to the inertial force of the lumped mass. Moreover, at either the node point (mode node) or peak point of the ith mode (Fig. 2), the design sensitivity of the ith natural frequency equals zero, which means that the ith frequency reaches an extremum. As is well known, when a lumped mass attachment occurs at a node point of the ith mode shape, it makes no impact on the natural frequency and the associated vibration mode. In this case, the frequency reaches its maximum value, that is, its "original" value without the lumped mass attachment. For example, a lumped mass attached at a rigid support (simply or clamped) of the beam would not affect any of the natural frequencies. If it appears at a peak point of the ith mode shape, a lumped mass will cause a maximal reduction in that frequency, and the frequency attains its minimum value. Thus, if one wishes to introduce a lumped mass to bring about the maximal effect on a specific natural frequency, the mass should be attached to a peak point of the corresponding mode shape.

Thus far, the preceding derivation is valid only for distinct natural frequencies. For repeated frequencies, Mills-Curran[9] carried out the sensitivity calculation by solving a subeigenvalue problem. The method can be used for repeated frequency sensitivity calculations as to the lumped mass position without any difficulties.

B. Plate Element with a Lumped Mass Attachment

In this section, the preceding work is extended to a thin-plate element, that is, the classical Kirchhoff flexural plate element with an attached lumped mass. In Fig. 3, the transverse displacement of the lumped mass M along the z axis can be represented approximately in terms of the nodal displacements and slopes:

$$w(a, b) = [N]_{(a,b)} \cdot \{\delta\}_e \tag{12}$$

where $[N]$ is a row vector of shape functions of the rectangular thin-plate element and $\{\delta\}_e$ is a column vector of element nodal degrees of freedom:

$$[N] = [N_1 \quad N_{x1} \quad N_{y1} \quad N_2 \quad N_{x2} \quad N_{y2} \quad N_3 \quad N_{x3} \quad N_{y3} \quad N_4 \quad N_{x4} \quad N_{y4}] \tag{13a}$$

$$\{\delta\}_e = [w_1 \quad \theta_{x1} \quad \theta_{y1} \quad w_2 \quad \theta_{x2} \quad \theta_{y2} \quad w_3 \quad \theta_{x3} \quad \theta_{y3} \quad w_4 \quad \theta_{x4} \quad \theta_{y4}]^T \tag{13b}$$

Similar to the earlier derivation, the equivalent mass matrix of a lumped mass is obtained by

$$[M]_L = M \begin{bmatrix} N_1^2 & N_1 N_{x1} & N_1 N_{y1} & \cdots & N_1 N_{x4} & N_1 N_{y4} \\ & N_{x1}^2 & N_{x1} N_{y1} & \cdots & N_{x1} N_{x4} & N_{x1} N_{y4} \\ & & N_{y1}^2 & \cdots & N_{y1} N_{x4} & N_{y1} N_{y4} \\ & & & \vdots & \vdots & \vdots \\ & \text{sym.} & & & & N_{y4}^2 \end{bmatrix}_{12 \times 12} \tag{14}$$

Therefore, the sensitivity of the ith natural frequency is obtained as follows:

$$\frac{\partial \omega_i^2}{\partial a} = -\omega_i^2 \{\phi_e\}_i^T \frac{\partial [M]_L}{\partial a} \{\phi_e\}_i \tag{15a}$$

$$\frac{\partial \omega_i^2}{\partial b} = -\omega_i^2 \{\phi_e\}_i^T \frac{\partial [M]_L}{\partial b} \{\phi_e\}_i \tag{15b}$$

Now we can take the standard shape functions of the thin-plate element[10] and utilize their characteristics at the element corner vertices. Assume that the lumped mass is attached at vertex 1 of the rectangular element (Fig. 3). After manipulations similar to those before, the sensitivity of the ith natural frequency is gained:

$$\left.\frac{\partial \omega_i^2}{\partial a}\right|_{\substack{a=0\\b=0}} = 2\omega_i^2 M w_{1i} \theta_{y1i} \tag{16a}$$

$$\left.\frac{\partial \omega_i^2}{\partial b}\right|_{\substack{a=0\\b=0}} = -2\omega_i^2 M w_{1i} \theta_{x1i} \tag{16b}$$

where w_{1i}, θ_{x1i}, and θ_{y1i} represent, respectively, the transverse displacement and slopes of the ith vibration mode along the x and y axes at vertex 1 of the plate element.

Likewise, the sensitivities of the lumped mass attachment at other corner vertices of the plate element can be conducted. Because of the compatibility of the nodal displacement quantities between two neighboring elements (though the nonconformity exists for a normal slope across the interface between two neighboring elements), the results obtained are identical to those of Eq. (16). Therefore, the subscripts indicating the element vertex in Eq. (16) will be omitted subsequently.

For the classical Kirchhoff flexural plate element, we can write the following relationships between the transverse displacements and slopes:

$$\theta_x = w_{,y} = \frac{\partial w}{\partial y}, \qquad \theta_y = -w_{,x} = -\frac{\partial w}{\partial x} \tag{17}$$

By substitution of the preceding expressions into Eq. (16), the sensitivity of the ith natural frequency is simplified as

$$\frac{\partial \omega_i^2}{\partial a} = -2\omega_i^2 M w_i w_{,xi} \tag{18a}$$

$$\frac{\partial \omega_i^2}{\partial b} = -2\omega_i^2 M w_i w_{,yi} \tag{18b}$$

where $w_{,xi}$ and $w_{,yi}$ are the partial derivatives of $w_i(x, y)$ to x and y axes, respectively, at the lumped mass attachment position (a, b) for the ith mode. Oguamanam et al.[2] have presented the same results for a Reissner–Mindlin plate element from Rayleigh's principle of stationary values.

Thus far, we have assumed that the lumped mass possibly moves along the edges of the element. If the element edge is not parallel to the global axes, the directional derivative can be calculated in a specific direction by the use of the gradient of a natural frequency:

$$\frac{d\omega_i^2}{ds} = \mathrm{grad}\left(\omega_i^2\right) \cdot \mathbf{ds} = \frac{\partial \omega_i^2}{\partial a}\cos\alpha + \frac{\partial \omega_i^2}{\partial b}\cos\beta \tag{19}$$

where $\{\cos\alpha, \cos\beta\}$ are the direction cosines of the specific direction in the global coordinate system.

III. Frequency Sensitivity Analysis with Respect to Lumped Mass Magnitude

For the purpose of completeness, the frequency sensitivity analysis with respect to the lumped mass magnitude is also described briefly in this section. Assume that a lumped mass M_n is rigidly attached to the node t of an FE model of a structure. It contributes to the global mass matrix as follows:

$$[M] = \begin{bmatrix} \cdots & & \vdots & & \vdots \\ & m_t^x + M_n & & & \\ & & m_t^y + M_n & & \\ & \vdots & & m_t^z + M_n & \\ \vdots & & & & \cdots \end{bmatrix} \tag{20}$$

where m_t^x, m_t^y, and m_t^z are the original structural transverse inertias related to node t. Then, the design sensitivity of the ith natural frequency is obtained:

$$\frac{d\omega_i^2}{dM_n} = -\omega_i^2\{\phi\}_i^T \frac{d[M]}{dM_n}\{\phi\}_i = -\omega_i^2 \cdot \left(u_t^2 + v_t^2 + w_t^2\right)_i \tag{21}$$

where $\{u_t, v_t, w_t\}_i$ are the transverse displacements of the lumped mass at node t, corresponding to the ith mode shape.

As expected, the frequency sensitivity to the lumped mass magnitude is always negative in Eq. (21), which means that the lumped mass magnitude always has an inverse effect on the natural frequencies. Increasing the magnitude would generally decrease the structural frequencies, and vice versa. However, when a lumped mass is attached at a node point of a vibration mode, its design sensitivity becomes zero, and the change of its magnitude would not alter the corresponding frequency. This is consistent with that of a lumped mass position. When a lumped mass is attached at a peak point of a vibration mode, the frequency sensitivity to its magnitude is the most negative locally, which is different from that to its position at the same position completely. Therefore, if one desires to reduce the magnitude of a lumped mass from a vibration system to increase a specific natural frequency, it is most efficient to reduce the lumped mass at the peak points of the related mode. Moreover, with these sensitivity features of a lumped mass, one can adjust the structural frequency alternatively.

Furthermore, note that the design sensitivity of the natural frequency in Eq. (21) does not involve the magnitude of a lumped mass explicitly. In fact, the natural frequency ω_i and the orthogonally normalized $\{\phi\}_i$ usually depend on the lumped mass magnitude. Therefore, the sensitivity of the natural frequency is associated with the magnitude through both the frequency and the amplitude of the lumped mass in the corresponding mode.

IV. Optimization Procedure for Lumped Mass Position

In frequency optimization problems with respect to lumped mass position, more than one lumped mass is often attached to the structure. Usually, the total structural weight is fixed. Positions of lumped masses, not their magnitudes within this context, are referred to as design variables. These variables change discretely because the lumped mass is assumed to be attached only at the nodes of the structural FE model. The structure is subject to natural frequency constraints to avoid resonance with the external excitations. In addition, the lumped mass may be restricted within an acceptable domain. In the design of aerospace vehicles, the center of mass of a structure is an important design consideration. Therefore, the optimization problem is formulated as follows:

Minimize movement of the lumped masses

Subject to

$$\omega_i \in \Omega_i^* \qquad (i = 1, 2, \ldots, n) \tag{22}$$

$$a_j \in D_j \qquad (j = 1, \ldots, k) \tag{23}$$

where a_j indicates the jth lumped mass position. Ω_i^* is the predefined limitation for the ith frequency, D_j the prescribed design region within which the jth lumped mass should be located, n the number of the constrained frequencies, and k the possibly movable lumped masses.

In this paper, an evolutionary optimization method[6,7] is employed to optimize the positions of the lumped masses because this algorithm is more suitable for treating a discrete optimization problem. The method consists of two steps. The first step is to find the most efficient lumped mass on the basis of design sensitivity analysis. The second step is to relocate the lumped mass to improve the constrained frequencies. The design sensitivity obtained in Sec. II, rather than the FE difference scheme (due to its higher computational costs) is utilized to estimate the design effect. Lumped masses are shifted along the element edges by the element size. The direction of movement is determined according to the sensitivity analysis. Therefore, to avoid a large change in the location of the structural center of mass, the lumped mass with the highest efficiency should be identified and moved to the neighboring node in each optimization loop. The optimization process continues until the frequencies satisfy the prescribed limits or the design variables reach their bounds. A block diagram of the process is shown in Fig. 4.

Following are the outlines of the optimization procedure for various frequency requirements[6,7]:

1) To increase or decrease a specified natural frequency ω_i, it is desirable to shift a lumped mass with the maximum absolute value of

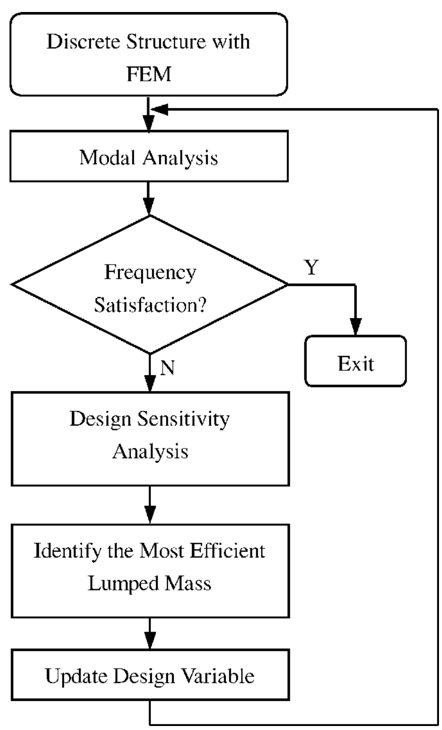

Fig. 4 Flowchart for evolutionary optimization algorithm.

Table 1 Natural frequencies (hertz) and sensitivities of the lumped mass attachment

Number	Original frequency	Node 7[a] Frequency	Node 7[a] Sensitivity	Node 6[a] Frequency	Node 6[a] Sensitivity	Node 5[a] Frequency	Node 5[a] Sensitivity	Node 4[a] Frequency	Node 4[a] Sensitivity
1	4.59	3.99	0.0	4.03	$-2.388E+1$	4.12	$-4.628E+1$	4.26	$-6.269E+1$
2	18.36	18.36	0.0	17.80	$1.618E+3$	16.80	$1.427E+3$	16.22	$3.112E+2$
3	41.30	36.92	0.0	38.94	$-1.268E+4$	41.30	0.0	39.28	$1.103E+4$

[a]With lumped mass attachment.

the design sensitivity among all of those that are potentially movable:

$$\max\left\{\left|\frac{\partial\omega_i^2}{\partial a_j}\right|,\quad j=1,k\right\} \tag{24}$$

The frequency change can be expressed linearly in terms of the variable variation:

$$\Delta\omega_i^2 \approx \frac{\partial\omega_i^2}{\partial a_j}\cdot\Delta a_j \tag{25}$$

Thus, to increase the natural frequency ω_i^2, one could get the following relationship for the direction of movement of the lumped mass:

$$\mathrm{sign}(\Delta a_j) = \mathrm{sign}\left(\frac{\partial\omega_i^2}{\partial a_j}\right) \tag{26}$$

where Δa_j is the move interval of the lumped mass and sign() is the sign function.

Conversely, the direction of movement of a lumped mass for decreasing a specified natural frequency ω_i^2 is

$$\mathrm{sign}(\Delta a_j) = -\mathrm{sign}\left(\frac{\partial\omega_i^2}{\partial a_j}\right) \tag{27}$$

2) To increase two specific natural frequencies ω_i and ω_m simultaneously, it is desirable to change the position of the lumped mass with the maximum absolute value of the sum of the design sensitivities corresponding to ω_i^2 and ω_m^2 among all of the possibly shifted ones. That is,

$$\max\left\{\left|\frac{\partial\omega_i^2}{\partial a_j}+\frac{\partial\omega_m^2}{\partial a_j}\right|,\quad j=1,k\right\} \tag{28}$$

3) To increase the gap (space) between two neighboring natural frequencies $\omega_{i+1}-\omega_i$, it is desirable to change the position of the lumped mass with the maximum absolute value of the difference of the design sensitivities corresponding to ω_{i+1}^2 and to ω_i^2. That is,

$$\max\left\{\left|\frac{\partial\omega_{i+1}^2}{\partial a_j}-\frac{\partial\omega_i^2}{\partial a_j}\right|,\quad j=1,k\right\} \tag{29}$$

4) To increase a specified natural frequency ω_i while simultaneously keeping another one, ω_m, unchanged, it is desirable to shift such a lumped mass with the greater absolute sensitivity for ω_i^2 along with the smaller absolute value of the sensitivity for ω_m^2 among all of the possibly movable ones. That is,

$$\max\left\{\left|\frac{\partial\omega_i^2}{\partial a_j}\right|\Big/\left|\frac{\partial\omega_m^2}{\partial a_j}\right|,\quad j=1,k\right\} \tag{30}$$

Furthermore, the direction of movement of the lumped mass is determined by

$$\mathrm{sign}(\Delta a_j) = \mathrm{sign}\left(\frac{\partial\omega_i^2}{\partial a_j}\right) \tag{31}$$

To implement the optimization procedure, the FE method is utilized to calculate the natural frequencies and the associated modes.

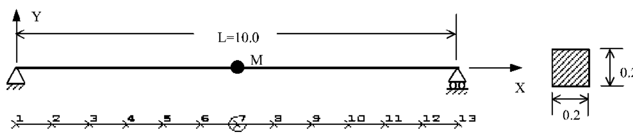

Fig. 5 Simply supported uniform beam and its FE model.

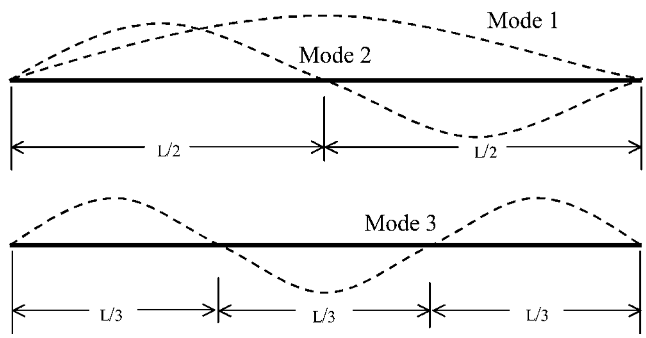

Fig. 6 Theoretical mode shapes of a simply supported uniform beam.

Then, nodal displacements and slopes at the lumped mass attachments are extracted, and then the frequency sensitivities can be achieved immediately. To reduce computational costs of modal analysis and sensitivity calculations, sometimes more than one lumped mass with the higher efficiency is shifted in an optimization loop.

Note that for a general structural system, the frequency is no longer a convex function to the position of a lumped mass because the frequency may attain its maximum with the attachment at any node point, or its minimum at any peak point of the related mode. Therefore, the frequency optimization, in this case, is no longer a convex programming problem. To obtain global optimal design, the optimization process has to be started from several different initial designs.

V. Illustrative Examples

The described frequency sensitivity and the evolutionary algorithm are applied to frequency optimization of three structures in terms of the positions of the lumped masses. The program is developed for multiple frequency constraints based on the commercial software SAMCEF®/Dynam. A consistent element mass matrix is adopted in structural modal analysis.

A. Simply Supported Beam

This simple example aims at illustrating the frequency sensitivity and optimization process schematically. A uniform beam of length $L=10$ m, with a lumped mass $M=500$ kg at the midspan, is simply supported at both ends. The beam is discretized equally with 12 elements and is shown in Fig. 5. The cross section is a square with its side $H=0.2$ m. Young's modulus is $E=2.1\times10^{11}$ Pa and material density $\rho=7800$ kg/m^3. The first three original vibration modes without the lumped mass attachment are shown in Fig. 6. The associated frequencies are tabulated in the first column of Table 1. Frequencies with the lumped mass attachment at the midspan and the corresponding design sensitivities are tabulated in the second and third column of Table 1 for comparison. Just as expected, the first and third natural frequencies decrease significantly because

Table 2 Basic data (millimeters) of the free–free beam

Segment	Length	Thickness	Diameter
1	550	4.0	140
2	550	3.0	210
3	700	4.5	210
4	200	3.0	210
5	570	5.5	210
6	640	3.5	210
7	390	3.0	210
8	360	3.0	210

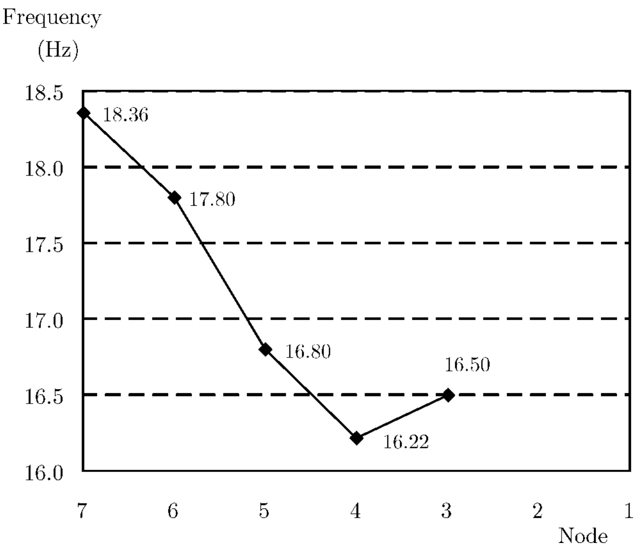

Fig. 7 Evolutionary process of the second frequency for its descent.

the lumped mass is located exactly at the peak point of the related modes, whereas the second frequency is insensitive to the lumped mass because it is located exactly at the node point of the associated mode. In addition, note that all of the design sensitivities are zeros.

First, we intend to raise the third frequency to its original value. The lumped mass is finally moved to node 5 where the frequency sensitivity reaches zero. Solutions are listed in the sixth column of Table 1.

Second, we wish to decrease the second frequency to its minimum value. The lumped mass starts from node 7 and finally reaches node 4. Frequencies are listed in the eighth column in Table 1. The frequency evolution with respect to the positions of the lumped mass is shown in Fig. 7. Note that the left peak point of the second mode shape, in this case, moves into the element between node 3 (1.667 m) and node 4 (2.5 m). Therefore, the design sensitivity of the lumped mass is positive at node 4, rather than zero. However, the frequency becomes higher when the lumped mass moves to node 3, and the corresponding sensitivity is negative ($-1.235E+3$). In fact, the lumped mass may oscillate between these two nodes. The exact minimum of the second frequency is 16.19 Hz, with the lumped mass attached at 2.318 m away from the left end.

This example demonstrates the validity of the frequency sensitivity analysis and illustrates the optimization process. In addition, it is verified again that when a lumped mass is attached at a node point of a specific mode, it has no effect on the corresponding frequency. Otherwise, if a lumped mass were not attached at the node points of a mode, it would affect both the frequency value and the associated mode shape.

B. Free–Free Beam

In this example, a free–free beam is optimized for its natural frequencies with some lumped masses movable and some fixed at specified nodes. The beam has a tubular cross section and is modeled with 30 beam elements uniformly discretized in each segment, as shown in Fig. 8. The basic data are listed in Tables 2 and 3,

Table 3 Nodes attached with fixed and movable lumped masses

Lumped masses, kg	Fixed lumped masses, nodes	Movable lumped masses	
		Initial nodes	Optimal nodes
2	3, 4	3, 7, 11, 15, 18, 23, 27, 30	9, 9, 10, 15, 18, 24, 27, 27
4	5–8	——	——
5	21–31	——	——
7	9–20	——	——

Table 4 Initial and final frequencies (hertz) with the locations of the center of mass (meters)

Frequency	Initial	Final
1	34.26	35.72
2	88.30	90.74
Center of mass	2.164	2.172

Table 5 Variation of element thickness (millimeters) along the longitudinal direction of the wing

Thickness	Elements
40	1–40
30	41–70
20	71–100

Table 6 Frequencies (hertz) of the wing with and without the lumped masses

Mode	Original	Initial	Optimal
Bending	2.58	2.56	2.10
Torsional	11.76	11.31	11.76
Gap	9.18	8.75	9.66

respectively. Assume that eight movable lumped masses are initially attached at the middle of each segment. The flexural frequencies and the structural center of mass are listed in Table 4. Note that there are three rigid-body mode shapes in the modal analysis. Young's modulus is $E = 7.0 \times 10^{10}$ Pa and material density $\rho = 2700\,\mathrm{kg/m^3}$. The center of mass is only allowed to shift by ± 0.02 m, and the first two flexural frequencies are required to be within the limits of 36 ± 0.5 and 91.5 ± 1 Hz, respectively. The total mass is 201.6 kg.

The evolutionary histories of the constrained frequencies are illustrated in Fig. 9. The optimal positions of the eight movable lumped masses are listed in Table 3 for comparison with their initial designs. The first two frequencies meet the prescribed limitations, and the structural center of mass shifts by 0.008 m, as seen in Table 4.

C. Wing

The preliminary model of a sweep wing is discretized regularly by 100 Kirchhoff plate elements. The structure is made up of aluminum with Young's modulus $E = 7.0 \times 10^{10}$ Pa, Poisson ratio $\nu = 0.3$, and material density $\rho = 2700\,\mathrm{kg/m^3}$. Nodes on the wing root (left edge) are all fixed. The element thickness varies along the longitudinal direction of the wing, as listed in Table 5. Original frequencies corresponding to the first bending mode (the first frequency) and the first torsional mode (the third one) are tabulated in the first column of Table 6. To raise the flutter speed of the airplane, an important measure is to maximize the gap between the first torsional and the first bending frequencies by adding dynamic balances. Initially, five lumped masses $M = 6$ kg are distributed in the structural domain, as shown in Fig. 10. The related mode shapes are illustrated by contour lines in Figs. 11 and 12, respectively. The corresponding frequencies are listed in the second column of Table 6. At the optimal design, all of the lumped masses are located at the trailing edge of the wing tip, point A, because this point is the maximum amplitude of the bending mode and is on the nodal line of the torsional mode. The optimal

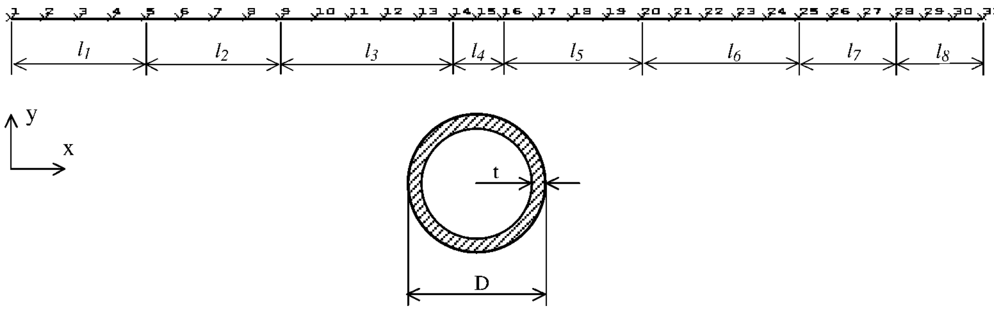

Fig. 8 FE model of a free–free beam.

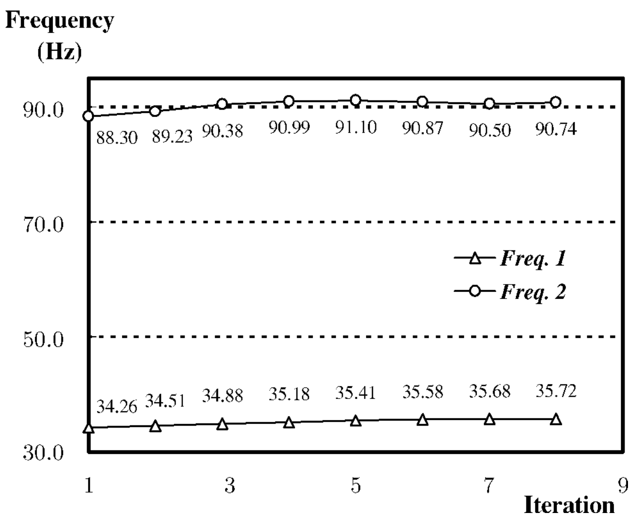

Fig. 9 Evolutionary histories of the constrained frequencies.

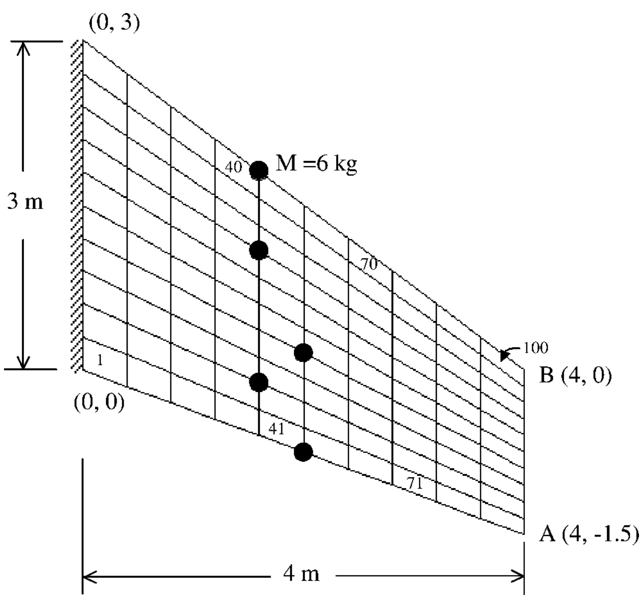

Fig. 10 Preliminary model of a sweep wing with lumped mass attachments.

solutions are listed in the third column of Table 6 for comparison. It is obvious that, at the optimum, the torsional frequency is not affected by the presence of the lumped masses. Figure 13 illustrates the evolutionary histories of the redesigned frequencies and their differences, respectively. However, when the aeroelastic requirement is taken into account in practical design, the lumped masses are usually located at the leading edge of the wing tip, point B. In this case, the flutter speed might decline somewhat.

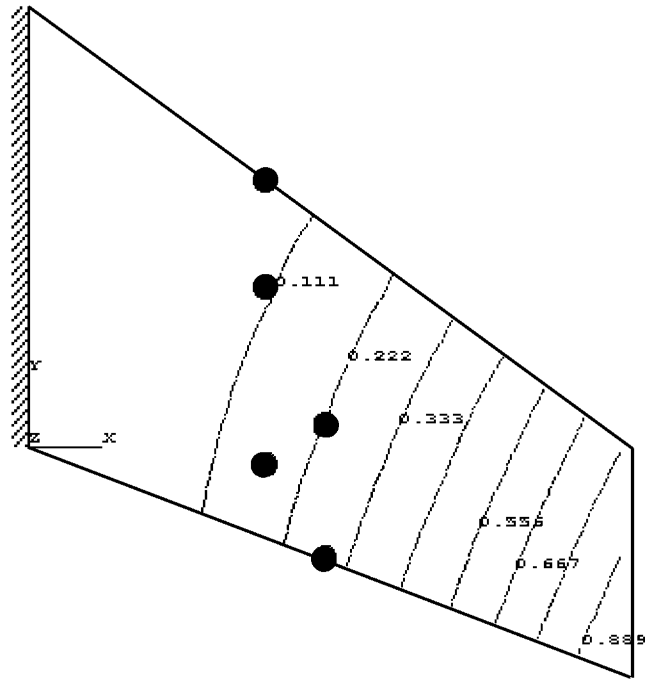

Fig. 11 First bending mode shape of the wing.

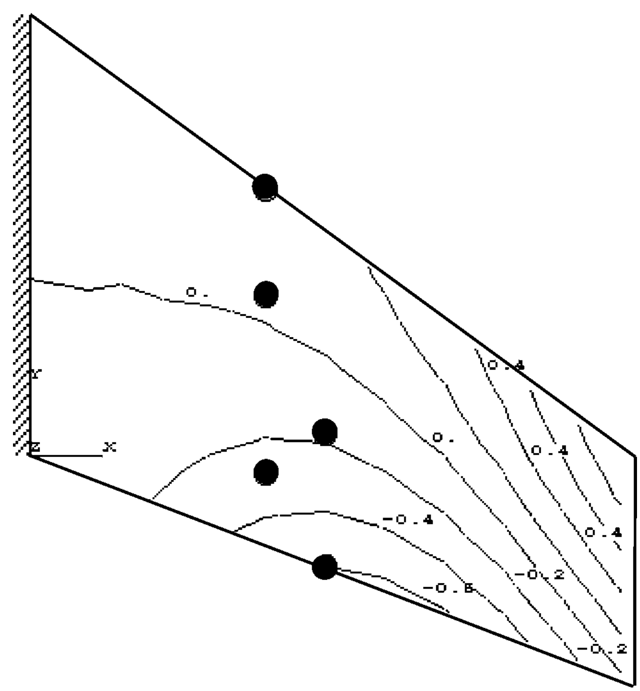

Fig. 12 First torsional mode shape of the wing.

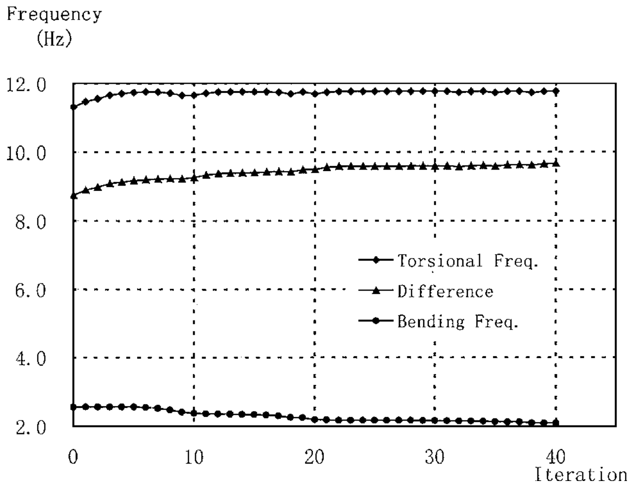

Fig. 13 Evolutionary histories of the redesigned frequencies and their difference.

VI. Conclusions

In this paper, the frequency sensitivity analysis is investigated with respect to the position and magnitude of a lumped mass by the discrete method, and the same formulas are obtained in agreement with previous studies. It turns out that the frequency sensitivity with respect to the position of a lumped mass is proportional to its inertial force. When a lumped mass is attached at the node points of a specific mode, all of the frequency sensitivities to the position and magnitude are zeros, and the lumped mass does not affect the frequency. Based on the sensitivity analysis, an evolutionary algorithm is presented for frequency optimization of beam or plate structures as to the positions of the lumped mass attachments. Three examples illustrate the effectiveness and validity of the proposed method.

In this paper, only the transverse inertia of each lumped mass is studied, whereas their rotatory inertia is ignored. In practical design, it is more accurate to consider both the transverse and rotatory inertias of a lumped mass simultaneously.

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